Time to Financial Independence, Accounting for Expenses and/or Income Growing Faster or Slower Than Inflation

Engineering Your FI

DERIVATION

This document derives the number of years n to achieve financial independence (FI) when expenses or income grow faster or slower than inflation.

Savings each year S are defined in terms of income I and expenses E:

$$S = I - E \tag{1}$$

The standard method for computing time to FI accounts for inflation by assuming income and expenses increase each year by exactly the amount of inflation, and assumed market returns are simply lowered by the average long term inflation rate (e.g. a nominal long term average market return rate of 10% is lowered to 7% to account for long term average inflation of 3%). Thus all values remain in "current day dollars".

But what if expenses grow more slowly, or more quickly, than inflation? Similarly, what if income grows more slowly, or more quickly, than inflation?

To account for these varying growth rates, savings each year can be written as

$$\begin{aligned}
S_0 &= I - E \\
S_1 &= I(1 + R_I) - E(1 + R_E) \\
S_2 &= I(1 + R_I)^2 - E(1 + R_E)^2 \\
&\dots \\
S_n &= I(1 + R_I)^n - E(1 + R_E)^n
\end{aligned} \tag{2}$$

where R_I is the difference in the rate of your income change each year versus inflation (e.g. if you get a 4% raise each year with 3% inflation, $R_I = 1\%$), and R_E is the difference in the rate of your expenses change each year versus inflation (e.g. if your expenses increase 2% each year with 3% inflation, $R_E = -1\%$).

To determine when FI is achieved, the final value for expenses $E_F = E(1+R_E)^n$ is set equal to the safe withdrawal rate WR (e.g. 0.04 if using the 4% rule) times the future portfolio value FV needed for FI (with the standard assumption that expenses will increase by exactly the inflation rate each year after achieving FI):

$$E_F = E(1 + R_E)^n = \text{WR FV} \tag{3}$$

FV can be broken up into two components:

$$E(1 + R_E)^n = \text{WR}(\text{FVA} + \text{FVB}) \tag{4}$$

where FVA is the future value from the initial value IV (amount of money you start with), equal to

$$\text{FVA} = \text{IV}(1 + R)^n \tag{5}$$

where R is the investment interest rate AFTER accounting for inflation, and FVB is the future value from ongoing contributions S_i (savings each year).

To compute FVB, let's start by writing it as the sum of each year's contribution multiplied by the appropriate appreciation factor to account for the growth of that contribution over time:

$$\begin{aligned} \text{FVB} &= S_1(1 + R)^{n-1} + S_2(1 + R)^{n-2} + \cdots + S_n(1 + R)^0 \\ &= \sum_{i=1}^{n} S_i(1 + R)^{n-i} \end{aligned} \tag{6}$$

where $S_1(1 + R)^{n-1}$ is the first contribution after the first time interval (year), which thus has one less year to appreciate, and $S_n(1 + R)^0 = S_n$ is the final contribution at the end date, which thus has no time to appreciate.

The expression for the savings values S_i found in equation 2 can be substituted into equation 6 to obtain

$$\begin{aligned} \text{FVB} &= \sum_{i=1}^{n} \left(I(1 + R_I)^i - E(1 + R_E)^i\right)(1 + R)^{n-i} \\ &= \sum_{i=1}^{n} I(1 + R_I)^i(1 + R)^{n-i} - \sum_{i=1}^{n} E(1 + R_E)^i(1 + R)^{n-i} \\ &= I(1 + R)^n \sum_{i=1}^{n} \left(\frac{1 + R_I}{1 + R}\right)^i - E(1 + R)^n \sum_{i=1}^{n} \left(\frac{1 + R_E}{1 + R}\right)^i \end{aligned} \tag{7}$$

The FVB consists of two geometric series terms, which we can derive closed form solutions for. To make that derivation easier, the following temporary substitutions are made:

$$\begin{aligned}
\text{FVB} &= B \\
(1+R)^n &= X \\
\frac{1+R_I}{1+R} &= Y \\
\frac{1+R_E}{1+R} &= Z
\end{aligned} \tag{8}$$

so that equation 7 can be written as

$$B = IX\sum_{i=1}^{n} Y^i - EX\sum_{i=1}^{n} Z^i \tag{9}$$

The first geometric series can be reformulated as:

$$\begin{aligned}
C &= IX\sum_{i=1}^{n} Y^i \\
&= IXY + IXY^2 + \cdots + IXY^n \\
CY &= IXY^2 + \cdots + IXY^n + IXY^{n+1} \\
C - CY &= IXY - IXY^{n+1} \\
C(1-Y) &= IXY(1-Y^n) \\
C &= IXY\frac{1-Y^n}{1-Y}
\end{aligned} \tag{10}$$

And the second can be addressed similarly:

$$\begin{aligned}
D &= EX\sum_{i=1}^{n} Z^i \\
&= EXZ\frac{1-Z^n}{1-Z}
\end{aligned} \tag{11}$$

Substituting equations 10 and 11 into 9, we get

$$B = IXY\frac{1-Y^n}{1-Y} - EXZ\frac{1-Z^n}{1-Z} \tag{12}$$

Then substituting the expressions in 8 into 12, we get

$$\text{FVB} = I(1+R)^n\frac{1+R_I}{1+R}\left[\frac{1-\left(\frac{1+R_I}{1+R}\right)^n}{1-\left(\frac{1+R_I}{1+R}\right)}\right] - E(1+R)^n\frac{1+R_E}{1+R}\left[\frac{1-\left(\frac{1+R_E}{1+R}\right)^n}{1-\left(\frac{1+R_E}{1+R}\right)}\right] \tag{13}$$

Whew, that's quite an equation! Let's see if we can simplify:

$$
\begin{aligned}
\text{FVB} &= I(1+R)^n(1+R_I)\left[\frac{1-\left(\frac{1+R_I}{1+R}\right)^n}{(1+R)-(1+R_I)}\right] - E(1+R)^n(1+R_E)\left[\frac{1-\left(\frac{1+R_E}{1+R}\right)^n}{(1+R)-(1+R_E)}\right] \\
&= I(1+R_I)\left[\frac{(1+R)^n-(1+R_I)^n}{R-R_I}\right] - E(1+R_E)\left[\frac{(1+R)^n-(1+R_E)^n}{R-R_E}\right] \\
&= I\frac{1+R_I}{R-R_I}\left[(1+R)^n-(1+R_I)^n\right] - E\frac{1+R_E}{R-R_E}\left[(1+R)^n-(1+R_E)^n\right]
\end{aligned} \tag{14}
$$

To check this equation, plug in zero for all R_I and R_E terms to confirm the result matches the standard expression for a future value given ongoing contributions S with returns rate R:

$$
\begin{aligned}
\text{FVB} &= I\frac{1+0}{R-0}\left[(1+R)^n-(1+0)^n\right] - E\frac{1+0}{R-0}\left[(1+R)^n-(1+0)^n\right] \\
&= I\frac{(1+R)^n-1}{R} - E\frac{(1+R)^n-1}{R} \\
&= S\frac{(1+R)^n-1}{R}
\end{aligned} \tag{15}
$$

Important note: there are two singularities in the above approach. The first singularity is when $R = R_I$ (i.e. market returns exactly equal income growth rate), which produces a zero in the denominator of the first term in equation 14. The second singularity is when $R = R_E$ (i.e. market returns exactly equal expense growth rate), which produces a zero in the denominator of the second term in equation 14. These singularities are addressed in Appendix A: When Market Returns Rate Equals Income Growth Rate and Appendix B: When Market Returns Rate Equals Expense Growth Rate. The even more rare situation in which both singularities occur simultaneously ($R = R_E = R_I$) is discussed in Appendix C: When Expense Growth Rate and Income Growth Rate Equals Market Returns Rate.

Now both FVA and FVB from equations 5 and 14 can be plugged into equation 4:

$$
E(1+R_E)^n = \text{WR}\left[\text{IV}(1+R)^n + I\frac{1+R_I}{R-R_I}\left[(1+R)^n-(1+R_I)^n\right] - E\frac{1+R_E}{R-R_E}\left[(1+R)^n-(1+R_E)^n\right]\right] \tag{16}
$$

and dividing both sides of equation 16 by $(1+R_E)^n$:

$$
E = \text{WR}\left[\text{IV}\left(\frac{1+R}{1+R_E}\right)^n + I\frac{1+R_I}{R-R_I}\left(\frac{1+R}{1+R_E}\right)^n - I\frac{1+R_I}{R-R_I}\left(\frac{1+R_I}{1+R_E}\right)^n - E\frac{1+R_E}{R-R_E}\left(\frac{1+R}{1+R_E}\right)^n + E\frac{1+R_E}{R-R_E}\right] \tag{17}
$$

and attempting to solve for n:

$$
E - \text{WR}E\frac{1+R_E}{R-R_E} = \text{WR}\left(\text{IV} + I\frac{1+R_I}{R-R_I} - E\frac{1+R_E}{R-R_E}\right)\left(\frac{1+R}{1+R_E}\right)^n - \text{WR}I\frac{1+R_I}{R-R_I}\left(\frac{1+R_I}{1+R_E}\right)^n \tag{18}
$$

Setting all expressions other than n to constant terms:

$$a = bc^n - de^n \tag{19}$$

one can see that unfortunately there is no direct analytical solution for n.

Thus to solve for n, a simple and fast optimization method known as the Newton Rhapson method is employed. First the equation 19 is re-arranged to have an expression set equal to zero, as the Newton Rhapson method is a root-finding method:

$$0 = bc^n - de^n - a \tag{20}$$

This expression is the function employed in the Newton Rhapson method:

$$f(x) = bc^x - de^x - a \tag{21}$$

The derivative of that expression is:

$$f'(x) = \log(c)\, bc^x - \log(e)\, de^x \tag{22}$$

The Newton Rhapson method loops over the equation

$$x_{m+1} = x_m - \frac{f(x)}{f'(x)} \tag{23}$$

until $x_{m+1} - x_m$ is less than some provided threshold (e.g. 0.001). The initial value x_0 for the loop is set equal to the number of years to FI with both R_I and R_E set to zero, as found in the Savings Rate math spec.

$$x_0 = \ln\left(\frac{1 - \theta + \frac{\text{WR}\ \theta}{R}}{\frac{\text{WR IV}}{I} + \frac{\text{WR}\ \theta}{R}}\right) / \ln(1 + R) \tag{24}$$

where θ is the savings rate (S/I).

APPENDIX A: WHEN MARKET RETURNS RATE EQUALS INCOME GROWTH RATE

If $R = R_I$, the Y value in equation 8 equals 1, and thus the geometric series described in 10 is:

$$\begin{aligned} C &= IX \sum_{i=1}^{n} Y^i \\ &= IX \sum_{i=1}^{n} 1 \\ &= IXn \\ &= I(1+R)^n n \end{aligned} \tag{25}$$

Plugging this new formulation for C into equation 9 and then equations 13 and 14, we get:

$$\text{FVB} = I(1+R)^n n - E\frac{1+R_E}{R-R_E}\left[(1+R)^n - (1+R_E)^n\right] \tag{26}$$

Plugging this new formulation for FVB into 4:

$$E(1+R_E)^n = \text{WR}\left[\text{IV}(1+R)^n + I(1+R)^n n - E\frac{1+R_E}{R-R_E}\left[(1+R)^n - (1+R_E)^n\right]\right] \tag{27}$$

and dividing both sides of equation 27 by $(1+R_E)^n$:

$$E = \text{WR}\left[\text{IV}\left(\frac{1+R}{1+R_E}\right)^n + In\left(\frac{1+R}{1+R_E}\right)^n - E\frac{1+R_E}{R-R_E}\left(\frac{1+R}{1+R_E}\right)^n + E\frac{1+R_E}{R-R_E}\right] \tag{28}$$

and attempting to solve for n:

$$E - \text{WR}E\frac{1+R_E}{R-R_E} = \text{WR}\left(\text{IV} + In - E\frac{1+R_E}{R-R_E}\right)\left(\frac{1+R}{1+R_E}\right)^n \tag{29}$$

Setting all expressions other than n to constant terms:

$$a = (b+cn)d^n \tag{30}$$

one can see that unfortunately there is no direct analytical solution for n.

Thus to solve for n, the numerical Newton Rhapson method is employed as described above. A numerical Lambert W function method may also be employed, but the Newton Rhapson method is employed for consistency.

The functions employed for the Newton Rhapson method are:

$$\begin{aligned} f(x) &= (b + cx)d^x - a \\ f'(x) &= cd^x + \log(d)(b + cx)d^x \end{aligned} \tag{31}$$

APPENDIX B: WHEN MARKET RETURNS RATE EQUALS EXPENSE GROWTH RATE

If $R = R_E$, the Z value in equation 8 equals 1, and thus the geometric series described in 10 is:

$$\begin{aligned} D &= EX \sum_{i=1}^{n} Z^i \\ &= EX \sum_{i=1}^{n} 1 \\ &= EXn \\ &= E(1+R)^n n \end{aligned} \tag{32}$$

Plugging this new formulation for D into equation 9 and then equations 13 and 14, we get:

$$\text{FVB} = I \frac{1+R_I}{R-R_I} \left[(1+R)^n - (1+R_I)^n\right] - E(1+R)^n n \tag{33}$$

Plugging this new formulation for FVB into 4:

$$E(1+R_E)^n = \text{WR}\left[\text{IV}(1+R)^n + I\frac{1+R_I}{R-R_I}\left[(1+R)^n - (1+R_I)^n\right] - E(1+R)^n n\right] \tag{34}$$

and dividing both sides of equation 34 by $(1+R_E)^n = (1+R)^n$:

$$E = \text{WR}\left[\text{IV} + I\frac{1+R_I}{R-R_I} - I\frac{1+R_I}{R-R_I}\left(\frac{1+R_I}{1+R}\right)^n - En\right] \tag{35}$$

and attempting to solve for n:

$$E - \text{WR IV} - \text{WR } I\frac{1+R_I}{R-R_I} = -\text{WR } I\frac{1+R_I}{R-R_I}\left(\frac{1+R_I}{1+R}\right)^n - \text{WR } En \tag{36}$$

Setting all expressions other than n to constant terms:

$$a = bc^n - dn \tag{37}$$

one can see that unfortunately there is no direct analytical solution for n.

Thus to solve for n, the numerical Newton Rhapson method is employed as described above. A numerical Lambert W function method may also be employed, but the Newton Rhapson method is employed for consistency.

The functions employed for the Newton Rhapson method are:

$$\begin{aligned} f(x) &= bc^x - dx - a \\ f'(x) &= \log(c)bc^x - d \end{aligned} \tag{38}$$

APPENDIX C: WHEN EXPENSE GROWTH RATE AND INCOME GROWTH RATE EQUALS MARKET RETURNS RATE

In the rare situation where the expense growth rate R_E, income growth rate R_I, and investment returns rate R are all exactly equal, both terms of the FVB expression in equation 7 are simplified as shown in equations 25 and 32, resulting in relatively simple expression for FVB:

$$\text{FVB} = I(1+R)^n n - E(1+R)^n n \tag{39}$$

Plugging this new formulation for FVB into 4:

$$E(1+R_E)^n = \text{WR}\left[\text{IV}(1+R)^n + I(1+R)^n n - E(1+R)^n n\right] \tag{40}$$

and dividing both sides of equation 40 by $(1+R_E)^n = (1+R)^n$:

$$E = \text{WR IV} + \text{WR } In - \text{WR } En \tag{41}$$

and solving for n:

$$n = \frac{E - \text{WR IV}}{\text{WR}(I - E)} \tag{42}$$