Time To Financial Independence

Engineering Your FI

DERIVATION

This document derives the number of years n to achieve financial independence (FI). To start, your savings each year S is defined in terms of your income I and your expenses E:

$$S = I - E \tag{1}$$

Equation 1 is then re-arranged for expenses E and we set expenses equal to the assumed withdrawal rate WR needed for FI (e.g. 0.04 if using the 4% rule) times the future portfolio value FV needed for FI:

$$E = I - S = \text{WR FV} \tag{2}$$

FV can be broken up into two components:

$$I - S = \text{WR}(\text{FVA} + \text{FVB}) \tag{3}$$

where FVA is the future value from the initial value IV (amount of money you start with), equal to

$$\text{FVA} = \text{IV}(1 + R)^n \tag{4}$$

where R is the investment interest rate, and FVB is the future value from ongoing contributions S (savings each year), equal to

$$\text{FVB} = S\frac{(1 + R)^n - 1}{R} \tag{5}$$

Both of these future value equations are well established in literature*, but the expression for FVB is also derived in the Appendix A: Ongoing Contribution Derivation.

*Wikipedia, Compound Interest, Monthly Deposits

Plugging FVA and FVB from equations 4 and 5 into equation 3:

$$I - S = \text{WR}\left(\text{IV}(1+R)^n + S\frac{(1+R)^n - 1}{R}\right) \tag{6}$$

and dividing both sides of equation 6 by income I:

$$\begin{aligned} 1 - \theta &= \text{WR}\ \frac{\text{IV}}{I}\ (1+R)^n + \text{WR}\ \theta\ \frac{(1+R)^n - 1}{R} \\ &= \frac{\text{WR IV}}{I}\ (1+R)^n + \frac{\text{WR}\ \theta}{R}\ (1+R)^n - \frac{\text{WR}\ \theta}{R} \end{aligned} \tag{7}$$

where θ is the savings rate. Rearranging and solving for n:

$$\begin{aligned} 1 - \theta + \frac{\text{WR}\ \theta}{R} &= \left(\frac{\text{WR IV}}{I} + \frac{\text{WR}\ \theta}{R}\right)(1+R)^n \\ (1+R)^n &= \frac{1 - \theta + \frac{\text{WR}\ \theta}{R}}{\frac{\text{WR IV}}{I} + \frac{\text{WR}\ \theta}{R}} \\ n &= \ln\left(\frac{1 - \theta + \frac{\text{WR}\ \theta}{R}}{\frac{\text{WR IV}}{I} + \frac{\text{WR}\ \theta}{R}}\right) / \ln(1+R) \end{aligned} \tag{8}$$

If the initial value IV of your portfolio is zero, the expression for n simplifies to:

$$\begin{aligned} n &= \ln\left(\frac{1 - \theta + \frac{\text{WR}\ \theta}{R}}{\frac{\text{WR}\ \theta}{R}}\right) / \ln(1+R) \\ &= \ln\left(\frac{1}{\text{WR}\ \theta}\ (R - R\theta) + 1\right) / \ln(1+R) \end{aligned} \tag{9}$$

which is function only of the assumed withdrawal rate WR, the investment interest rate R, and your savings rate θ.

To rewrite the number of years n until FI in terms of expenses E and savings rate θ, instead of income I, first the expression for θ must be re-arranged to solve for I:

$$\begin{aligned} \theta &= \frac{S}{I} = \frac{I - E}{I} = 1 - \frac{E}{I} \\ \frac{E}{I} &= 1 - \theta \\ I &= \frac{E}{1 - \theta} \end{aligned} \tag{10}$$

Then the expression for I in equation 10 is plugged into equation 8:

$$\begin{aligned} n &= \ln\left(\frac{1 - \theta + \frac{\text{WR}\ \theta}{R}}{\frac{\text{WR IV}}{E/(1-\theta)} + \frac{\text{WR}\ \theta}{R}}\right) / \ln(1+R) \\ &= \ln\left(\frac{1 - \theta + \frac{\text{WR}\ \theta}{R}}{\frac{\text{WR IV}\ (1-\theta)}{E} + \frac{\text{WR}\ \theta}{R}}\right) / \ln(1+R) \end{aligned} \tag{11}$$

Thus for a given initial portfolio value IV, the lower your expenses E are, the lower the number of years n to FI is. In other words, for a given absolute IV, the lower your expenses are, the closer you are to reaching your FI number.

APPENDIX A: ONGOING CONTRIBUTION DERIVATION

To derive the expression for the future value of an ongoing contribution S as provided in equation 5, FVB is first written as the sum of all the contributions with the appreciation of each contribution by the relevant time:

$$\begin{aligned} \text{FVB} &= S\left((1+R)^{n-1} + (1+R)^{n-2} + \cdots + (1+R)^0\right) \\ &= \sum_{i=1}^{n} S(1+R)^{n-i} \end{aligned} \tag{12}$$

where $S(1+R)^{n-1}$ is the first contribution after the first time interval (year), which thus has one less year to appreciate, and $S(1+R)^0 = S$ is the final contribution at the end date, which thus has no time to appreciate.

FVB can be rewritten as a geometric series,

$$\begin{aligned} \text{FVB} &= S\sum_{i=1}^{n}(1+R)^n \frac{1}{(1+R)^i} \\ &= S(1+R)^n \sum_{i=1}^{n} \left(\frac{1}{1+R}\right)^i \end{aligned} \tag{13}$$

To temporarily simplify notation, $B =$ FVB, $X = S(1+R)^n$, $Y = \frac{1}{1+R}$. Thus equation 13 becomes

$$\begin{aligned} B &= X\sum_{i=1}^{n} Y^i \\ &= XY + XY^2 + \cdots + XY^n \end{aligned} \tag{14}$$

Multiplying B by Y:

$$BY = XY^2 + XY^3 + \cdots + XY^{n+1} \tag{15}$$

And then subtracting BY from B:

$$\begin{aligned} B - BY &= XY - XY^{n+1} \\ B(1-Y) &= XY(1-Y^n) \end{aligned} \tag{16}$$

Solving for $B =$ FVB and re-entering the original expressions:

$$\begin{aligned}
B &= XY\frac{1-Y^n}{1-Y} \\
FVB &= S(1+R)^n \frac{1}{1+R}\left(\frac{1-\left(\frac{1}{1+R}\right)^n}{1-\left(\frac{1}{1+R}\right)}\right) \\
&= S(1+R)^n \left(\frac{1-\left(\frac{1}{1+R}\right)^n}{1+R-1}\right) \\
&= S\frac{(1+R)^n-1}{R}
\end{aligned} \tag{17}$$