Time To Financial Independence, Starting With Debt

Engineering Your FI

DERIVATION

This document derives the number of years n to achieve financial independence (FI) for someone starting with debt.

To start, let's derive the time it takes to pay off the loan.

Note: given that nearly all loans require monthly payments, we will compute "payoff time" in terms of months and then convert that to years for the overall Time to FI calculation.

The amount you pay off of the loan each month is

$$\Delta P = s - P_0 r \tag{1}$$

where P_0 is the initial value of the loan (also known as the initial principal), r is the monthly loan interest rate (typically computed by lenders as dividing the annual rate R_l (otherwise known as the APR) by 12 (even though that's not mathematically correct), and s is how much savings you have each year, which we will assume goes entirely towards paying off the loan (for simplicity - you may need to replenish emergency funds, etc.). You can see how the interest $P_0 r$ fights against your effort to pay off the loan with your payments s.

The loan balance, i.e. principal, after one month is thus

$$\begin{aligned} P_1 &= P_0 - \Delta P \\ &= P_0 - (s - P_0 r) \\ &= P_0(1 + r) - s \end{aligned} \tag{2}$$

If we continue this monthly payment pattern to subsequent months:

$$\begin{aligned} P_1 &= P_0(1 + r) - s \\ P_2 &= P_1(1 + r) - s \\ P_3 &= P_2(1 + r) - s \\ &\vdots \\ P_{m+1} &= P_m(1 + r) - s \end{aligned} \tag{3}$$

To find the expression for the final balance P_n, which will be zero for a paid off loan, let's first determine P_3 to more easily identify a recurring pattern:

$$
\begin{aligned}
P_3 &= (P_1(1+r) - s)(1+r) - s \\
&= P_1(1+r)^2 - s(1+r) - s \\
&= (P_0(1+r) - s)(1+r)^2 - s(1+r) - s \\
&= P_0(1+r)^3 - s(1+r)^2 - s(1+r) - s
\end{aligned} \tag{4}
$$

Studying P_3, a pattern emerges for P_n:

$$
P_n = P_0(1+r)^n - \left[s + s(1+r) + s(1+r)^2 + \cdots + s(1+r)^{n-1}\right] \tag{5}
$$

where the rightmost expression in brackets is a geometric series. Fortunately the derivation of the closed form solution of a geometric series is straightforward:

$$
\begin{aligned}
g &= s + s(1+r) + s(1+r)^2 + \cdots + s(1+r)^{n-1} \\
g(1+r) &= s(1+r) + s(1+r)^2 + \cdots + s(1+r)^{n-1} + s(1+r)^n \\
g - g(1+r) &= s - s(1+r)^n \\
g(1-(1+r)) &= s(1-(1+r)^n) \\
g &= s\frac{1-(1+r)^n}{-r}
\end{aligned} \tag{6}
$$

which can be plugged into equation 5 to obtain the closed form expression for P_n

$$
P_n = P_0(1+r)^n - s\frac{1-(1+r)^n}{-r} \tag{7}
$$

We can then set P_n to zero for a fully paid off loan, and solve for the number of months required to pay off the loan:

$$\begin{aligned}
0 &= P_0(1+r)^n - s\frac{1-(1+r)^n}{-r} \\
P_0(1+r)^n &= s\frac{(1+r)^n - 1}{r} \\
\frac{P_0 r}{s} &= \frac{(1+r)^n - 1}{(1+r)^n} \\
\frac{P_0 r}{s} &= 1 - \frac{1}{(1+r)^n} \\
(1+r)^{-n} &= 1 - \frac{P_0 r}{s} \\
-n\ln(1+r) &= \ln\left(1 - \frac{P_0 r}{s}\right) \\
n &= \frac{-\ln\left(1 - \frac{P_0 r}{s}\right)}{\ln(1+r)}
\end{aligned} \tag{8}$$

However, we are interested in knowing the time required to pay off a loan in terms of someone's saving rate and expenses. Thus we must convert the savings amount s to an expression consisting of monthly expenses e and savings rate θ (for which the monthly and yearly values are identical).

First, your savings each month s is defined in terms of your monthly income i and your monthly expenses e:

$$s = i - e \tag{9}$$

Your savings rate can be written as

$$\begin{aligned}
\theta &= \frac{s}{i} \\
&= \frac{i-e}{i} \\
&= 1 - \frac{e}{i}
\end{aligned} \tag{10}$$

which can then be isolated for i and substituted into equation 9:

$$\begin{aligned}
i &= \frac{e}{1-\theta} \\
s &= \frac{e}{1-\theta} - e \\
s &= e\left(\frac{1}{1-\theta} - 1\right)
\end{aligned} \tag{11}$$

This expression for s can be substituted into equation 8 to get the number of months needed to pay off the loan in terms of the initial loan balance P_0, monthly interest rate r, monthly expenses e, and savings rate θ:

$$n = \frac{-\ln\left(1 - \dfrac{P_0 r}{e\left(\dfrac{1}{1-\theta} - 1\right)}\right)}{\ln(1+r)} \tag{12}$$

This expression can be written in terms of annual quantities by substituting the monthly interest rate r with the annual interest rate (APR) R_l divided by 12, substituting the monthly expenses e value with the yearly expenses E value divided by 12, and dividing the number of months n by 12 to obtain the total number of years N to pay off the loan:

$$N_l = \frac{-\ln\left(1 - \dfrac{P_0 \dfrac{R_l}{12}}{\dfrac{E}{12}\left(\dfrac{1}{1-\theta} - 1\right)}\right)}{12\ \ln\left(1 + \dfrac{R_l}{12}\right)} \tag{13}$$

We can then combine this expression with the “Time to FI from \$0” expression from the “Time To FI” derivation document to obtain the total time to FI from a net worth less than \$0:

$$N = \frac{-\ln\left(1 - \dfrac{P_0 R_l}{E\left(1-\theta\right)^{-1} - E}\right)}{12\ \ln\left(1 + R_l/12\right)} + \frac{\ln\left(\dfrac{1}{\text{WR}\ \theta}\left(R - R\theta\right) + 1\right)}{\ln(1+R)} \tag{14}$$

where WR the assumed withdrawal rate and R is the investment interest rate.